

Selected Aspects of the Mathematical Work of Krzysztof P. Wojciechowski

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Dedicated to Krzysztof P. Wojciechowski on his 50th birthday

ABSTRACT. To honor and to please our friend Krzysztof P. Wojciechowski I will review the milestones of his mathematical work. This will at the same time be a tour of Analysis and Geometry of Boundary Value Problems. Starting in the 80s I will discuss the spectral flow and the general linear conjugation problem, the Calderón projector and the topology of space of elliptic boundary problems. The theme of the 90s is the eta invariant. The paper with Douglas was fundamental for establishing spectral invariants for manifolds *with* boundary and for the investigation of the behavior of spectral invariants under analytic surgery. This was so influential that many different proofs of the gluing formula for the eta-invariant were published. Finally turning to the new millennium we will look at the zeta-determinant. Compared to eta this is a much more rigid spectral invariant which is technically challenging.

1. Introduction

1.1. The framework and the problem. To begin with let us describe in general terms the problems to which Krzysztof P. Wojciechowski has contributed so much in the last 25 years.

Let X be a compact smooth Riemannian manifold with boundary $\Sigma = \partial X$. Furthermore, let E, F be hermitian vector bundles over X and let

$$(1) \quad D : \Gamma^\infty(X, E) \longrightarrow \Gamma^\infty(X, F)$$

be an elliptic differential operator: $\Gamma^\infty(X, E)$ denotes the spaces of smooth sections of the bundle E .

In this situation some natural questions occur:

1. *What are appropriate boundary conditions for D on X ?*

This question is absolutely fundamental since without imposing boundary conditions we cannot expect D to have any reasonable spectral theory.

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A boundary condition is given by a pseudo-differential operator

$$(2) \quad P : \Gamma^\infty(\Sigma, E) \longrightarrow \Gamma^\infty(\Sigma, E)$$

of order 0.¹ The *realization* D_P of the boundary condition given by P is the differential expression D acting on the domain

$$(3) \quad \text{dom}(D_P) := \{u \in L_1^2(X, E) \mid P(u|_\Sigma) = 0\}.$$

Since D is elliptic what one should expect naturally for P to be "appropriate" is that *elliptic regularity* holds. That is if $Du \in L_s^2(X, E)$ ² is of Sobolev order $s \geq 0$ and if $P(u|_\Sigma) = 0$ then $u \in L_{s+d}^2(X, E)$ is already of Sobolev order $s + d$, where d denotes the order of D .

2. *What is the structure of the space of all (nice) boundary conditions and how do spectral invariants of D_P depend on the boundary condition?*

These problems are the Leitfaden of Krzysztof P. Wojciechowski's work. If we are given a realization D_P of a nice boundary value problem we can do spectral theory and study the basic spectral invariants of D_P . We will see that the question in the headline leads to interesting and delicate analytical problems. Let us specify the kind of spectral invariants we mean here.

The most basic spectral invariant of the Fredholm operator D_P is its index

$$(4) \quad \text{ind } D_P = \dim \ker D_P - \dim \text{coker } D_P.$$

More rigid (and analytically more demanding) spectral invariants are derived from the heat trace

$$(5) \quad \text{tr}(e^{-tD_P^2}) = \sum_{\lambda \in \text{spec } D_P \setminus \{0\}} e^{-t\lambda^2},$$

where D_P is now assumed to be self-adjoint, via Mellin transforms. The most important examples are the η -invariant

$$(6) \quad \eta(D_P) = \left[\frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{(s-1)/2} \text{tr}(D_P e^{-tD_P^2}) dt \right]_{s=0}$$

and the ζ -determinant

$$(7) \quad \log \det_\zeta(D) = -\frac{d}{ds} \left[\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{-tD_P^2}) dt \right]_{s=0}.$$

The existence of these invariants is highly non-trivial since it depends on the meromorphic continuation of the right hand side of (6) and (7).

In the following sense the index is the least rigid and the ζ -determinant is the most rigid of these three invariants. In order not to get into too much technicalities assume for the moment that $D(s)_{a \leq s \leq b}$ is a smoothly varying family of elliptic operators on a closed manifold.³

¹One could think of more general definitely nonlocal boundary operators, but in this paper we will content ourselves to pseudo-differential boundary conditions.

²We denote the space of sections of E which are of Sobolev order s by $L_s^2(X, E)$.

³Here smoothly varying means that all coefficients depend smoothly on the parameter.

The index is insensitive to small perturbations of the operator. Hence $\text{ind } D(s)$ will not depend on s at all. The variation of the η -invariant is easy to understand. First of all the *reduced* η -invariant

$$(8) \quad \tilde{\eta}(D(s)) = \frac{1}{2}(\dim \ker D(s) + \eta(D(s)))$$

has only integer jumps and the total number of jumps equals the *spectral flow* of the family $D(s)$ over the interval $[a, b]$. The variation of $\tilde{\eta}(D(s)) \bmod \mathbb{Z}$ is *local* in the sense that $\frac{d}{ds}(\tilde{\eta}(D(s)) \bmod \mathbb{Z})$ is the integral of a density which is a local expression in terms of the coefficients of the operator and its derivatives, cf. Gilkey [12], Sec. 1.13. The variation of the ζ -determinant is more complicated and depends on global data.

It is therefore most natural that the early work of Krzysztof P. Wojciechowski dealt with problems related to the index. The paper [11] with Douglas is a landmark since it is the starting point of a whole decade seeing a lot of papers focusing on the η -invariant and the ζ -determinant. I was told that it came as an almost unbelievable surprise for the mathematical community when η -function and η -invariant for Dirac operators on compact manifolds *with* boundary were established in [11], since until then the η -invariant was only established for closed manifolds and considered solely as a natural *correction* term associated to index problems on manifolds with boundary and living exclusively on the boundary.

The paper [11] already contained one of the major analytical tools which has been refined and exploited ever since: the adiabatic method (see Section 3.1 below).

There is a variant of the problems mentioned above which I would like to point out. Suppose that M is a closed manifold which is *partitioned* by a separating hypersurface $\Sigma \subset M$. I. e. there are compact manifolds with boundary Y, X such that ⁴

$$(9) \quad M = Y \cup_{\Sigma} X.$$

After having chosen appropriate boundary conditions P^X, P^Y for D on X, Y we have three versions of D : D_{PX}, D_{PY} and the essentially self-adjoint operator D on the closed manifold M . In a sense we have " $D = D_{PY} \cup D_{PX}$ " and it is natural to ask how the spectral invariants of D, D_{PX} , and D_{PY} are related. Krzysztof P. Wojciechowski and his collaborators have provided us with spectacular results on this problem.

1.2. The basic framework of boundary value problems for Dirac type operators. Let us be a bit more specific now and describe the basic set-up of boundary value problems for Dirac type operators as we understand it today.

Let X and D be as before. We assume that D is an operator of *Dirac type*. That is in local coordinates

$$(10) \quad D^2 = -g^{ij}(x)I_{\text{rank } E} \frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order terms.}$$

⁴This is a situation which is typical for surgery theory in which we would have $\Sigma = S^k \times S^l, Y = S^k \times D^l$, where S^k denotes the unit sphere in \mathbb{R}^{k+1} and D^k denotes the unit disc in \mathbb{R}^k .

This is the most general notion of Dirac operator. The leading symbol of D

$$(11) \quad \sigma_D(x, df) = i[D, f]_x, \quad f \in C^\infty(X),$$

induces a *Clifford module* structure on E . That is we put for $v \in T_x X$ ⁵

$$(12) \quad c(v) := \frac{1}{i} \sigma_D(x, v^\flat).$$

Then $c(v)^2 = -g(v, v)$ and hence by the universal property of the Clifford algebra, c extends to a section of the bundle $\text{Hom}(Cl(TX, g), \text{End } E)$ of algebra-homomorphisms between the bundle of Clifford-algebras $Cl(TX, g)$ and the endomorphism bundle $\text{End } E$. This gives E the structure of a Clifford-module.

If we choose a Riemannian connection ∇ on E we can form the Dirac operator D^∇ on E which is locally given by

$$(13) \quad D^\nabla = \sum g^{ij} c\left(\left(\frac{\partial}{\partial x_i}\right)^\flat\right) \nabla_{\frac{\partial}{\partial x_j}}.$$

In the terminology of Booß-Bavnbek and Wojciechowski [7] such operators are called “generalized Dirac operators”. The operators D^∇ and D obviously have the same leading symbol, hence

$$(14) \quad D = D^\nabla + V$$

with $V \in \Gamma^\infty(X, \text{End } E)$.

Next we have to take the boundary of X into account. We fix a diffeomorphism from a collar U of the boundary onto $N := [0, \varepsilon) \times \Sigma$. Then we may choose a unitary transformation Φ from $L^2(U, E)$ onto the product Hilbert space $L^2([0, \varepsilon), L^2(\Sigma, E))$. The operator $\Phi D \Phi^{-1}$ which, by slight abuse of notation, will again be denoted by D then takes the form

$$(15) \quad D|N = J\left(\frac{d}{dx} + B(x)\right) + V(x)$$

where $J \in \Gamma^\infty(\Sigma, \text{End } E)$ is a unitary reflection ($J^2 = -I, J^* = -J$), $V \in C^\infty([0, \varepsilon), \Gamma^\infty(\Sigma, \text{End } E))$ and $(B(x))_{0 \leq x \leq \varepsilon}$ is a smooth family of first order formally self-adjoint differential operators on the closed manifold Σ (called the tangential operator).

Replacing $B(x)$ by $B(x) + J^{-1}V(x)$ we obtain alternatively

$$(16) \quad D|N = J\left(\frac{d}{dx} + \tilde{B}(x)\right)$$

at the expense that now $\tilde{B}(x)$ has only self-adjoint leading symbol.

We emphasize that J is independent of x and that (15) holds for all operators of Dirac type (Brüning and Lesch [9], Lemma 1.1). The representation (15) of a generalized Dirac operator is crucial for the geometry of their boundary value problems. In the existing literature, one could sometimes get the impression that for (15) to hold one needs that D is the Dirac operator of a Riemannian connection on E as in (13) or even a compatible Dirac operator.

⁵The Riemannian metric provides us with the “musical” isomorphisms $\flat : T_x M \rightarrow T_x^* M$ and $\sharp = \flat^{-1}$.

Furthermore, for many results to be presented below only the following properties of D will be needed:

- (1) D is first order formally self-adjoint elliptic,
- (2) D has the form (15) near the boundary,
- (3) D has the *unique continuation property*.

Properties of Dirac operators which are related to Clifford algebras will more or less play no role.

D is formally self-adjoint. That is for sections $f, g \in \Gamma^\infty(X, E)$ we have Green's formula

$$(17) \quad (Df, g) - (f, Dg) = - \int_{\Sigma} \langle f, g \rangle_{E_x} \, \text{dvol}(x).$$

In order to obtain an unbounded self-adjoint operator in $L^2(X, E)$ we have to impose appropriate boundary conditions.

For a pseudo-differential orthogonal projection⁶

$$P : L^2(\Sigma, E) \rightarrow L^2(\Sigma, E)$$

we define D_P to be the differential expression D acting on the domain (3).

DEFINITION 1.1. 1. *In the notation of (15) we abbreviate $B_0 := B(0)$ and denote by $P_+(B_0)$ the orthogonal projection onto the positive spectral subspace of B_0 . This is a pseudo-differential operator of order 0. Its principal symbol is denoted by $\sigma_{P_+(B_0)}$.*

2. *The boundary condition defined by P is called well-posed if for each $\xi \in T_x^*\Sigma \setminus \{0\}$ the principal symbol $\sigma_P(\xi)$ of P maps $\text{range } \sigma_{P_+(B_0)}(\xi)$ bijectively onto $\text{range } \sigma_P(\xi)$.*

This is Seeley's definition of well-posedness [22]. If P is well-posed then D_P has nice properties.

PROPOSITION 1.1. *Let P be well-posed. Then D_P is a Fredholm operator with compact resolvent. Moreover it is regular in the sense that if a distributional section u of E satisfies $Du \in L_s^2(X, E)$ and $P(u|_\Sigma) = 0$ then $u \in L_{s+1}^2(X, E)$, $s \geq 0$.*

It turns out that for Dirac type operators this notion of regularity already characterizes the class of well-posed boundary conditions as was shown by Brüning and Lesch [9].

So far we have basically presented the status of affairs from the point of view of classical elliptic theory.

2. The early work on spectral flow and the general linear conjugation problem

[25, 4, 5]

The early papers [25, 4, 5] (in part with Booß) on the general linear conjugation problem are fundamental for our today's understanding of the structure of boundary value problems of Dirac type operators. The linear

⁶This is not a big loss of generality. It can be shown that if the boundary operator has closed range then the boundary condition may be represented by an orthogonal projection

conjugation problem is the natural generalization of the classical Riemann Hilbert problem to elliptic operators (cf. [7], Sec. 26).

Consider a partitioned manifold $M = Y \cup_{\Sigma} X$ as in (9) and let

$$(18) \quad D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

be a super-symmetric Dirac operator. That is the bundle $E = E^+ \oplus E^-$ is \mathbb{Z}_2 -graded and D is odd with respect to this grading.

In a collar $N = (-\varepsilon, \varepsilon) \times \Sigma$ of Σ we write D in the form (16) and hence we get for D_+

$$(19) \quad D_+ = \sigma\left(\frac{d}{dx} + B(x)\right)$$

where $\sigma \in \Gamma^\infty(\Sigma, \text{Hom}(E^+, E^-))$ is unitary (and independent of x) and $(B(x))_{-\varepsilon \leq x \leq \varepsilon}$ is a smooth family of elliptic differential operators with self-adjoint leading symbol.

Furthermore, let $\Phi \in \Gamma^\infty(\Sigma, \text{Aut}(E))$ be a unitary bundle automorphism⁷ of E which is even with respect to the grading. Multiplication by Φ is a pseudo-differential operator of order 0 which we denote by the same letter. We assume that Φ commutes with the leading symbol of $B(x)$. As a consequence the operator $\Phi B \Phi^{-1} - B$ is of order 0 and $\Phi P_+(B(x)) - P_+(B(x))\Phi$ is of order -1 and thus acts as a compact operator on $L^2(\Sigma, E^+)$.

We introduce a local boundary value problem by letting the differential expression D_+ act on

$$(20) \quad \text{dom}(D_+^\Phi) := \{(u_1, u_2) \in L_1^2(Y, E^+) \oplus L_1^2(X, E^+) \mid u_1|_\Sigma = \Phi u_2|_\Sigma\}.$$

From Green's formula (17) one derives

$$(21) \quad (D_+^\Phi)^* = D_-^{\sigma\Phi\sigma^*}$$

and thus

$$(22) \quad D^{\Phi \oplus \sigma\Phi\sigma^*} = \begin{bmatrix} 0 & D_-^{\sigma\Phi\sigma^*} \\ D_+^\Phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & (D_+^\Phi)^* \\ D_+^\Phi & 0 \end{bmatrix}.$$

One can show that $D^{\Phi \oplus \sigma\Phi\sigma^*}$ is a realization of a local elliptic boundary value problem. Introducing the *Cauchy data spaces*

$$(23) \quad N(D_+, X) := \{u|_\Sigma \mid u \in L_{1/2}^2(\Sigma, E^+), D_+ u = 0\}$$

we find

$$(24) \quad \begin{aligned} \text{ind } D_+^\Phi &= \dim((\Phi N(D_+, X)) \cap N(D_-, Y)) \\ &\quad - \dim((J\Phi^* J^* N(D_-, X)) \cap N(D_-, Y)). \end{aligned}$$

Before we can state the main result on the linear conjugation problem we need to elaborate a bit more on the Cauchy data spaces.

⁷Krzysztof P. Wojciechowski originally treated more generally Φ 's which cover a diffeomorphism of Σ . Then multiplication by Φ is a Fourier integral operator.

2.1. Calderón projector and the smooth self-adjoint Grassmannian.

DEFINITION 2.1. *The (orthogonalized) Calderón projector $C(D, X)$ is the orthogonal projection onto the Cauchy data space $N(D, X)$.*

There is a little subtlety here. The natural construction of the Calderón projector via the invertible double (cf. [7], Sec. 12) gives a pseudo-differential (in general non-orthogonal) projection onto the Cauchy data space. It is an orthogonal projection if D is in product form (cf. (36) below) near the boundary. Of course, for any projection there is an orthogonal projection with the same image and using the results of Seeley [24] it follows that

PROPOSITION 2.1. *The orthogonalized Calderón projector $C(D, X)$ is a pseudo-differential operator of order 0. Its leading symbol coincides with the leading symbol $\sigma_{P_+(B_0)}$ of $P_+(B_0)$.*

The pseudo-differential properties of the Calderón projector had been developed by Calderón [10] and Seeley [23]. In [6] we will show that the orthogonalized Calderón projector can be constructed from a natural boundary value problem on the disconnected double $X \amalg X$. For brevity we will address the orthogonalized Calderón projector just as Calderón projector.

The in my view most important observation of the papers [4, 5] is the fact that the Cauchy data spaces are Lagrangian. To explain this note that on the Hilbert space $L^2(\Sigma, E)$ we have the symplectic form

$$(25) \quad \omega(f, g) := -(Jf, g).$$

This claim may be somewhat bewildering since $L^2(\Sigma, E)$ is firstly a complex vector space and secondly infinite-dimensional. Nevertheless, ω is a non-degenerate skew-adjoint sesqui-linear form and it turns out that it makes perfectly sense to talk about Lagrangians, symplectic reductions, Maslov indices etc. The only difference is that, due to the infinite-dimensionality, Fredholm conditions come into play. This is a fascinating story and an elaboration would definitely need more space. For some basics cf. Kirk and Lesch [14], Sec. 6. We state explicitly what Lagrangians are in $L^2(X, E)$.

LEMMA 2.1. *A subspace $L \subset L^2(X, E)$ is Lagrangian if and only if $L^\perp = J(L)$.*

The following is basically a consequence of Green's formula (17).

PROPOSITION 2.2. *A realization D_P of a boundary condition is a symmetric operator if and only if $\text{range } P$ is an isotropic subspace of $L^2(X, E)$. Moreover, if P is well-posed then D_P is self-adjoint if and only if $\text{range } P$ is Lagrangian.*

The following Theorem was proved first in [4]:

THEOREM 2.1. *Let X be a compact Riemannian manifold with boundary and let D be a Dirac type operator on X . Then the Cauchy data space of $N(D, X)$ is a Lagrangian subspace of $L^2(X, E)$ with respect to the symplectic structure (25) induced by Green's form.*

This theorem is not only beautiful. It is of fundamental importance. We are now able to describe spaces of well-posed boundary value problems as Grassmannian spaces:

DEFINITION 2.2. *Let \mathcal{P} be the space of all pseudo-differential orthogonal projections acting on $L^2(\Sigma, E)$.*

The pseudo-differential Grassmannian $\text{Gr}_1(B_0)$ is the space of $P \in \mathcal{P}$ such that

$$(26) \quad P - P_+(B_0) \text{ is of order } -1.$$

The space of $P \in \mathcal{P}$ such that the difference $P - P_+(B_0)$ is smoothing is denoted by $\text{Gr}_\infty(B_0)$.

Finally the self-adjoint (smooth) pseudo-differential Grassmannian $\text{Gr}_p^(B_0)$ is the space of $P \in \text{Gr}_p(B_0)$, $p \in \{1, \infty\}$, whose image is additionally Lagrangian.*

Since $P_+(B)$ and $C(D, X)$ have the same leading symbol (26) may be replaced by

$$(27) \quad P - C(D, X) \text{ is of order } -1.$$

Hence P and $C(D, X)$ also have the same leading symbol and thus it is obvious from the Definition 1.1 that the boundary condition given by P is well-posed.

Furthermore, since the difference of any two elements $P, Q \in \text{Gr}_1(B_0)$ is compact they form a *Fredholm pair*, that is

$$(28) \quad PQ : \text{range } Q \longrightarrow \text{range } P$$

is a Fredholm operator. The index of this Fredholm operator is denoted by $\text{ind}(P, Q)$. We have

$$(29) \quad \text{ind}(P, Q) = \dim(\ker P \cap \text{range } Q) - \dim(\text{range } P \cap \ker Q).$$

2.2. The main theorem on the general linear conjugation problem. We are now in a position to state the main result on the general linear conjugation problem.

THEOREM 2.2. *The index of the linear conjugation problem (20) is given by*

$$\begin{aligned} \text{ind } D^\Phi &= \text{ind}(I - C(D_+, Y), \Phi C(D_+, X) \Phi^{-1}) \\ &= \text{ind } D + \text{ind}(C(D_+, X) - \Phi C(D_+, Y)) \\ &= \text{ind } D + \text{ind}(P_+(B_0) - \Phi P_-(B_0)). \end{aligned}$$

There would be much more to say. This index theorem is related to a lot. It is a generalization of the classical Riemann Hilbert problem on the complex projective line. It is related to the spectral flow and to the index of generalized Toeplitz operators.

I will not go into that. But let me say that the papers [25, 4, 5] contain much more. They provide a comprehensive presentation of the spectral flow and its topological meaning, Fredholm pairs, and the construction of the Calderón projector. Also it is proved that $P_+(B_0)$ is a pseudo-differential operator.

3. The η -invariant

[11, 26, 27, 16, 28]

Let us start with some general remarks on η - and ζ -functions. Let T be an unbounded self-adjoint operator in the Hilbert space H . Assume that T has compact resolvent such that the spectrum of T consists of a sequence of eigenvalues

$$|\lambda_1| \leq |\lambda_2| \leq \dots \text{ (repeated according to their finite multiplicity)}$$

with $|\lambda_n| \rightarrow \infty$. If λ_n satisfies a growth condition

$$(30) \quad |\lambda_n| \geq Cn^\alpha,$$

for some $\alpha > 0$ then we can form the holomorphic functions

$$(31) \quad \begin{aligned} \eta(T; s) &:= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{(s-1)/2} \operatorname{tr}(Te^{-tT^2}) dt \\ &= \sum_{\lambda \in \operatorname{spec} T \setminus \{0\}} |\lambda|^{-s} \operatorname{sign} \lambda \\ &= \operatorname{tr}(T|T|^{-s-1}), \quad \operatorname{Re} s > \frac{1}{\alpha}, \end{aligned}$$

and

$$(32) \quad \begin{aligned} \zeta(T; s) &:= \sum_{\lambda \in \operatorname{spec} T \setminus \{0\}} \lambda^{-s} \\ &= \operatorname{tr}(T^{-s}), \quad \operatorname{Re} s > \frac{1}{\alpha}. \end{aligned}$$

If T is non-negative then $\zeta(T; s)$ is also a Mellin transform similar to the first equality in (31)

$$(33) \quad \zeta(T; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}(e^{-tT^2}) dt.$$

For general T the function $\zeta(T; s)$ can still be expressed in terms of Mellin transforms using the formula

$$(34) \quad \zeta(T; s) = \frac{1}{2}(\zeta(T^2; s/2) + \eta(T; s)) + e^{-i\pi s} \frac{1}{2}(\zeta(T^2; s/2) - \eta(T; s)).$$

Up to a technical point the existence of a short time asymptotic expansion of $\operatorname{tr}(Te^{-tT^2})$, $\operatorname{tr}(e^{-tT^2})$ and the meromorphic continuation of the functions $\zeta(T; s)$, $\eta(T; s)$ is equivalent (cf. Brüning and Lesch [8], Lemma 2.2, for the precise statement).

If T is an elliptic operator on a closed manifold then it follows from the celebrated work of Seeley [24] that $\eta(T; s)$, $\zeta(T; s)$ extend meromorphically to \mathbb{C} with a precise description of the location of the poles and their residues.

If $\eta(T; s)$ is meromorphic at least in a half plane containing 0 one defines the η -invariant of T as

$$(35) \quad \begin{aligned} \eta(T) &:= \frac{1}{2\pi i} \oint_{|s|=\varepsilon} \frac{\eta(T; s)}{s} ds \\ &= \text{constant term in the Laurent expansion at } 0 \\ &=: \eta(T; 0). \end{aligned}$$

In many situations one can even show that $\eta(T; s)$ is regular at 0. The η -invariant was introduced in the celebrated work of Atiyah, Patodi and Singer [1, 2, 3] as a boundary correction term in an index formula for manifolds with boundary.

We return to manifolds with boundary and consider again a compact Riemannian manifold X with boundary $\partial X = \Sigma$ and a formally self-adjoint operator of Dirac type acting on the hermitian vector bundle E .

From now on we assume that D is in *product form* near the boundary. That is in the collar $N = [0, \varepsilon) \times \Sigma$ of the boundary D takes the form

$$(36) \quad D|_N = J\left(\frac{d}{dx} + B\right)$$

with J, B as in (15) and such that B is independent of x . The formal self-adjointness of D and B then implies

$$(37) \quad JB + BJ = 0.$$

The next Theorem guarantees the existence of the η -invariant and the ζ -determinant on the smooth self-adjoint Grassmannian.

THEOREM 3.1. [28] *For $P \in \text{Gr}_\infty^*(B)$ the functions $\eta(D_P; s), \zeta(D_P; s)$ extend meromorphically to a half-plane containing 0 with poles of order at most 1. Furthermore, 0 is not a pole and $\zeta(D_P; 0)$ is independent of P .*

Let me say a few words about the strategy of proof. As pointed out before we have to prove short time asymptotic expansions for $\text{tr}(D_P e^{-tD_P^2})$ and $\text{tr}(e^{-tD_P^2})$. Duhamel's principle⁸ allows to separate the interior contributions and the contributions coming from the boundary. Namely, let $\varphi \in C_0^\infty([0, \varepsilon))$ be a cut-off function with $\varphi \equiv 1$ near 0. Extend φ by 0 to a smooth function on X .

Let \tilde{D} be any elliptic extension of D to a closed manifold⁹ and let $D_{P,0}$ be the model operator $J\left(\frac{d}{dx} + B\right)$ on the cylinder $[0, \infty) \times \Sigma$ with boundary condition P at $\{0\} \times \Sigma$. Then

$$(38) \quad \begin{aligned} \text{tr}(D_P e^{-tD_P^2}) &= \text{tr}(\varphi D_{P,0} e^{-tD_{P,0}^2}) + \\ &\quad \text{tr}((1 - \varphi)\tilde{D} e^{-t\tilde{D}^2}) + O(t^K), \quad t \rightarrow 0+ \end{aligned}$$

for any $K > 0$.

By local elliptic analysis the second term in (38) has a short time asymptotic expansion [12], Lemma 1.9.1. So one is reduced to the treatment of the model operator $D_{P,0}$. For the Atiyah–Patodi–Singer problem $P = P_+(B)$ there are explicit formulas for $e^{-tD_{P_+,0}^2}$ from which the asymptotic expansion can be derived using classical results on special functions. Finally, for $P \in \text{Gr}_\infty^*(B)$ the operator $D_{P,0}$ can be treated as a perturbation of the APS operator $D_{P_+,0}$ [28].

A completely different approach by Grubb [13] leads to the generalization of Theorem 3.1 to all well-posed boundary value problems.

⁸A big word for something very simple: the method of variation of the constant for first order inhomogeneous ordinary differential equations.

⁹The existence of such a \tilde{D} is not essential for the following result but it simplifies the exposition. For Dirac type operators we can choose \tilde{D} to be the invertible double.

3.1. The adiabatic limit. Let us explain the result of [11, 26, 27] on the adiabatic limit of the η -invariant. We start with a partitioned manifold $M = Y \cup_{\Sigma} X$. Then we stretch the neck by putting

$$\begin{aligned} X_R &= [0, R] \times \Sigma \cup_{\{R\} \times \Sigma} X, \\ Y_R &= [-R, 0] \times \Sigma \cup_{\{-R\} \times \Sigma} Y, \\ M_R &= Y_R \cup_{\{0\} \times \Sigma} X_R. \end{aligned}$$

Denote by $\tilde{\eta}(D, M_R)$ the reduced η -invariant of D on M_R and by $\tilde{\eta}(D_P, X_R)$ the reduced η -invariant of D_P on X_R .

THEOREM 3.2. *We have*

$$(39) \quad \begin{aligned} \lim_{R \rightarrow \infty} \tilde{\eta}(D, M_R) &\equiv \lim_{R \rightarrow \infty} \tilde{\eta}(D_{I-P_+(B)}, Y_R) \\ &+ \lim_{R \rightarrow \infty} \tilde{\eta}(D_{P_+(B)}, X_R) \pmod{\mathbb{Z}}. \end{aligned}$$

We should be a bit more specific about the meaning of $P_+(B)$ here. The positive spectral projection of B is Lagrangian if and only if B is invertible. If B is not invertible then one has to fix a Lagrangian subspace of the null space of B . So whenever a Lagrangian is needed we choose $P_+(B)$ such that

$$1_{(0, \infty)}(B) \leq P_+(B) \leq 1_{[0, \infty)}(B).$$

That this is possible follows from the Cobordism Theorem (cf. [11] or Lesch and Wojciechowski [16]).

In [11] it was shown that the η -invariant makes sense for generalized Atiyah–Patodi–Singer boundary conditions, i.e. for $D_{P_+(B)}$. Moreover, it was shown that $\lim_{R \rightarrow \infty} \tilde{\eta}(D_{P_+(B)}, X_R)$ exists. The limit can be interpreted as the η -invariant of the operator D on the manifold with cylindrical ends X_{∞} . The full strength of Theorem 3.2 was proved in [26, 27]. In fact the (mod \mathbb{Z} reductions) of the ingredients of formula (39) do not depend on \mathbb{Z} as was observed by W. Müller [18]. In this way we obtain the gluing formula for the η -invariant for the boundary condition $P_+(B)$. The following generalization to all $P \in \text{Gr}_{\infty}^*(B)$ is worked out in [28].

THEOREM 3.3. *Let $M = Y \cup_{\Sigma} X$ be a partitioned manifold and let D be a Dirac type operator which is in product form in a collar of Σ . Then for $P \in \text{Gr}_{\infty}^*(B)$*

$$(40) \quad \tilde{\eta}(D, M) \equiv \tilde{\eta}(D_P, X) + \tilde{\eta}(D_{I-P}, Y) \pmod{\mathbb{Z}}.$$

There is even a formula if $I - P$ is replaced by a general $Q \in \text{Gr}_{\infty}^*(-B)$. This is an extension of a formula for the variation of the η -invariant under a change of boundary condition from [16], cf. also Theorem 4.1 below.

Because of its importance let us look briefly at the method of proof.

The first observation is that the heat kernel of the model operator $D = J\left(\frac{d}{dx} + B\right)$ on the cylinder $\mathbb{R} \times \Sigma$ is explicitly known since D^2 is just a direct sum of one-dimensional Laplacians $-\frac{d^2}{dx^2} + b^2$. Let $\mathcal{E}_{\text{cyl}}(t; x, y)$ be this cylinder heat kernel. Furthermore, denote by $\mathcal{E}_R(t; x, y)$ the heat kernel of D on the stretched manifold M_R .

Next one chooses R -dependent cut-off functions $\phi_{j,R}, \psi_{j,R}, j = 1, 2$, as follows:

$$\psi_{2,R}(x) = \begin{cases} 0 & \text{if } |x| \leq 3R/7, \\ 1 & \text{if } |x| \geq 4R/7, \end{cases}$$

$$\psi_{1,R} = 1 - \psi_{2,R}.$$

Finally, choose $\phi_{j,R}$ such that $\phi_{j,R}\psi_{j,R} = \psi_{j,R}$. Then paste the heat kernel \mathcal{E}_R on M_R and the cylinder heat kernel to obtain the kernel

$$(41) \quad Q_R(t; x, y) = \phi_{1,R}(x)\mathcal{E}_{\text{cyl}}(t; x, y)\psi_{1,R}(y) + \phi_{2,R}(x)\mathcal{E}_R(t; x, y)\psi_{2,R}(y).$$

Then Duhamel's principle yields

$$(42) \quad \mathcal{E}_R(t) = Q_R(t) + \mathcal{E}_R \# C_R(t),$$

where $\#$ is a convolution and C_R is an error term.

It seems that not much is gained yet. The point is that Douglas and Wojciechowski [11] could show that in the adiabatic limit the error term is negligible in the following sense:

THEOREM 3.4. *There are estimates*

$$\|\mathcal{E}_R(t; x, y)\| \leq c_1 t^{-\dim X/2} e^{c_2 t} e^{-c_3 d^2(x, y)/t},$$

$$\|(\mathcal{E}_R \# C_R)(t; x, x)\| \leq c_1 e^{c_2 t} e^{-c_3 R^2/t}$$

with c_1, c_2, c_3 independent of R .

Note that this result is much more than e.g. (38). For the η - and ζ -determinant the full heat semigroup contributes. It is astonishing that nevertheless in the adiabatic limit the full integrals from 0 to ∞ in (6) and (7) split into contributions coming from the cylinder and from the interior of the manifold.

4. The relative η -invariant and the relative ζ -determinant

[16, 21]

Recall from Theorem 3.1 that for $P \in \text{Gr}_\infty^*(B)$ the ζ -function $\zeta(D_P; s)$ is regular at 0. One puts

$$(43) \quad \det_\zeta D_P := \begin{cases} \exp(-\zeta'(D_P; 0)), & 0 \notin \text{spec } D_P, \\ 0, & 0 \in \text{spec } D_P. \end{cases}$$

In view of (34) and Theorem 3.1 a straightforward calculation shows for D_P invertible

$$(44) \quad \det_\zeta D_P = \exp\left(i\frac{\pi}{2}(\zeta(D_P^2; 0) - \eta(D_P)) - \frac{1}{2}\zeta'(D_P^2; 0)\right).$$

We emphasize that the regularity of $\eta(D_P; s)$ and $\zeta(D_P; s)$ at $s = 0$ is essential for (44) to hold. (44) shows that the η -invariant is related to the phase of the ζ -determinant and that in general

$$(\det_\zeta D)^2 \neq \det_\zeta(D^2).$$

The natural question which arises at this point is

PROBLEM 4.1. *How does $\det_\zeta(D_P)$ depend on $P \in \text{Gr}_\infty^*(B)$?*

The answer to this problem has a long history. Since the only joint paper of Wojciechowski and myself deals with an aspect of the problem I take the liberty to add a few personal comments. In 1992 I was a Postdoc at University Augsburg. At that time the paper [11] had just appeared and the gluing formula for the η -invariant was in the air. Still much of our todays understanding of spectral invariants for Dirac type operators on manifolds with boundary was still in its infancy. When Gilkey visited he posed a special case of the Problem 4.1. If the tangential operator is not invertible there is no canonical Atiyah–Patodi–Singer boundary condition for D . The positive spectral projection of B is not in $\text{Gr}_\infty^*(B)$. Rather one has to choose a Lagrangian subspace $V \subset \ker B$ and put

$$P_V := 1_{(0,\infty)}(B) + \Pi_V,$$

where Π_V denotes the orthogonal projection onto V . Then $P_V \in \text{Gr}_\infty^*(B)$. The boundary condition given by P_V is called a generalized Atiyah–Patodi–Singer boundary condition. Gilkey asked how the eta-invariant depends on V .

I did some explicit calculations on a cylinder which let me guess the correct formula. However, I did not know how to prove it in general. Somewhat later Gilkey sent me a little note of Krzysztof dealing with the same problem. He urged us to work together. I was just a young postdoc and I felt honored that Krzysztof, whose papers I already admired, quickly agreed. Except writing papers with my supervisor this was my first mathematical collaboration. It was done completely by fax and email; Krzysztof and I met for the first time more than a year after the paper had been finished.

In [16] Krzysztof and I proved a special case of the following result. The result as stated is a consequence of the Scott–Wojciechowski Theorem as was shown in [14], Sec. 4. The Scott–Wojciechowski Theorem will be explained below.

THEOREM 4.1. *Let $P, Q \in \text{Gr}_\infty^*(B)$. Then*

$$(45) \quad \tilde{\eta}(D_P) - \tilde{\eta}(D_Q) \equiv \log \det_F(\Phi(P)\Phi(Q)^*) \pmod{\mathbb{Z}}.$$

If P or Q is the Calderón projector then (45) is even an equality [14].

The general answer to Problem 4.1 given by Scott and Wojciechowski [21] is just beautiful. To explain their result we need another bit of notation. Recall that J defines the symplectic form on $L^2(\Sigma, E)$ (25). Let

$$E = E_i \oplus E_{-i}$$

be the decomposition of E into the eigenbundles of J . If $P \in \text{Gr}_\infty^*(B)$ then

$$L = \text{range } P \subset L^2(\Sigma, E)$$

is Lagrangian and from Lemma 2.1 one easily infers that the restrictions of the orthogonal projections $\Pi_{\pm i} = \frac{1}{2i}(i \pm J)$ onto $E_{\pm i}$ map L bijectively onto $L^2(\Sigma, E_{\pm i})$ and

$$\Phi(P) := \Pi_{-i} \circ (\Pi_i|_L)^{-1}$$

is a unitary operator from $L^2(\Sigma, E_i)$ onto $L^2(\Sigma, E_{-i})$. For P we then have the formula

$$(46) \quad P = \frac{1}{2} \begin{pmatrix} I & \Phi(P)^* \\ \Phi(P) & I \end{pmatrix}.$$

For $P, Q \in \text{Gr}_\infty^*(B)$ the operator $\Phi(P)^*\Phi(Q) - I$ is smoothing and hence $\Phi(P)^*\Phi(Q)$ is of determinant class.

With these preparations, the Scott–Wojciechowski theorem reads as follows.

THEOREM 4.2. *Let $P \in \text{Gr}_\infty^*(B)$ and let $C(D, X)$ be the orthogonalized Calderón projector. Then*

$$(47) \quad \det_\zeta(D_P) = \det_\zeta(D_{C(D, X)}) \det_F\left(\frac{I + \Phi(C(D, X))\Phi(P)^*}{2}\right).$$

5. Adiabatic decomposition of the ζ -determinant

[15, 19, 20]

When the gluing formula for the η -invariant had been established it was Krzysztof's optimism that eventually lead to a similar result for the ζ -determinant. The author has to admit that he was an unbeliever: I could not see why a reasonable analytic surgery formula for the ζ -determinant should exist. Well, I was wrong. A fruitful collaboration of J. Park and Krzysztof P. Wojciechowski eventually proved that the adiabatic method, which originally had been developed in the paper [11], was even strong enough to prove an adiabatic surgery formula for the ζ -determinant.

Consider again the adiabatic setting M_R, X_R, Y_R as in (3.1). In order not to blow up the exposition too much I will not present the result in its most general form. Rather I will make the following technical assumptions:

- (1) The tangential operator B is invertible.
- (2) The L^2 -kernel of D on $X \cup [0, \infty) \times \Sigma$ and $Y \cup [0, \infty) \times \Sigma$ vanishes.

Then the adiabatic surgery theorems for the Laplacians read as follows:

THEOREM 5.1. *Let $\Delta_{\pm, R, d}$ be the Dirichlet extension of D^2 on X_R, Y_R resp.; D_R denotes the operator D on X_R . Then*

$$\lim_{R \rightarrow \infty} \frac{\det_\zeta D_R^2}{\det_\zeta \Delta_{+, R, d} \det_\zeta \Delta_{-, R, d}} = \sqrt{\det_\zeta B^2}.$$

THEOREM 5.2. *Let $D_{+, R, P_+}, D_{-, R, P_-}$ be the operator D with Atiyah–Patodi–Singer boundary conditions on X_R, Y_R resp. Then*

$$\lim_{R \rightarrow \infty} \frac{\det_\zeta D_R^2}{\det_\zeta D_{+, R, \Pi_+}^2 \det_\zeta D_{-, R, \Pi_-}^2} = 2^{-\zeta'(B^2, 0)}.$$

These technical assumptions mentioned above were removed in Park and Wojciechowski [20]. For details the reader should consult loc. cit.

Finally, the “adiabatic” results on the zeta-determinants obtained by Park and Wojciechowski are not adiabatic any more. Loya and Park [17] showed that most of those results (and more) are true without stretching. Krzysztof P. Wojciechowski did have different (and charming) ideas how to

remove stretching of the cylinders. Unfortunately, his serious illness did not allow him to fill all the details and finish the paper.

References

- [1] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
- [2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 405–432.
- [3] ———, *Spectral asymmetry and Riemannian geometry III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.
- [4] B. Booß and K. P. Wojciechowski, *Desuspension and splitting elliptic symbols I*, Ann. Global Anal. Geom. **3** (1985), 337–383.
- [5] ———, *Desuspension and splitting elliptic symbols II*, Ann. Global Anal. Geom. **4** (1986), 349–400.
- [6] B. Booß-Bavnbek, M. Lesch, and C. Zhu, *In preparation*.
- [7] B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser, Basel, 1993.
- [8] J. Brüning and M. Lesch, *On the eta-invariant of certain non-local boundary value problems*, Duke Math. J. **96** (1999), 425–468, dg-ga/9609001.
- [9] ———, *On boundary value problems for Dirac type operators: I. Regularity and self-adjointness*, J. Funct. Anal. **185** (2001), 1–62, math.FA/9905181.
- [10] A. Calderón, *Boundary value problems for elliptic equations*, Outlines of the joint Soviet–American symposium on partial differential equations (Novosibirsk), 1963, pp. 303–304.
- [11] R. G. Douglas and K. P. Wojciechowski, *Adiabatic limits of the η -invariants the odd-dimensional Atiyah–Patodi–Singer problem*, Comm. Math. Phys. **142** (1991), 139–168.
- [12] P. Gilkey, *Invariance theory, the heat equation, and the Atiyah–Singer index theorem*, 2. ed., CRC Press, Boca Raton, 1995.
- [13] G. Grubb, *Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems*, Ark. Mat. **37** (1999), 45–86.
- [14] P. Kirk and M. Lesch, *The η -invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16** (2004), 553–629, math.DG/0012123.
- [15] S. Klimek and K. P. Wojciechowski, *Adiabatic cobordism theorems for analytic torsion and η -invariant*, J. Funct. Anal. **136** (1996), 269–293.
- [16] M. Lesch and K. P. Wojciechowski, *On the η -invariant of generalized Atiyah–Patodi–Singer boundary value problems*, Illinois J. Math. **40** (1996), 30–46.
- [17] P. Loya and J. Park, *On the gluing problem for the spectral invariants of dirac operators*, To appear.
- [18] W. Müller, *Eta invariants and manifolds with boundary*, J. Differential Geom. **40** (1994), 311–377.
- [19] J. Park and K. P. Wojciechowski, *Adiabatic decomposition of the ζ -determinant of the Dirac Laplacian I. The case of invertible tangential operator*, Comm. Partial Differential Equations **27** (2002), 1407–1435, (with an Appendix by Y. Lee).
- [20] ———, *Scattering theory and adiabatic decomposition of the ζ -determinant of the Dirac Laplacian*, Math. Res. Lett. **9** (2002), 17–25.
- [21] S. G. Scott and K. P. Wojciechowski, *The ζ -determinant and Quillen determinant for a Dirac operator on a manifold with boundary*, Geom. Funct. Anal. **10** (1999), 1202–1236.
- [22] R. T. Seeley, *Topics in pseudo-differential operators*, C.I.M.E., Conference on pseudo-differential operators 1968, Edizioni Cremonese, Roma, 1969, pp. 169–305.
- [23] ———, *Singular integrals and boundary value problems*, Amer. J. Math. **88** (1966), 781–809.
- [24] ———, *Complex powers of an elliptic operator*, Proc. Sympos. Pure Math. **10** (1967), 288–307.

- [25] K. P. Wojciechowski, *Spectral flow and the general linear conjugation problem*, Simon Stevin **59** (1985), 59–91.
- [26] ———, *The additivity of the η -invariant. The case of an invertible tangential operator*, Houston J. Math. **20** (1994), 603–621.
- [27] ———, *The additivity of the η -invariant. The case of a singular tangential operator*, Comm. Math. Phys. **109** (1995), 315–327.
- [28] ———, *The ζ -determinant and the additivity of the η -invariant on the smooth, self-adjoint Grassmannian*, Comm. Math. Phys. **201** (1999), 423–444.

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